

# Averaging in cosmology

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## Abstract

In this paper we discuss the effect of local inhomogeneities on the global expansion of nearly FLRW universes, in a perturbative setting. We derive a generic linearized averaging operation for metric perturbations from basic assumptions, and we explicity the issue of gauge invariance. We derive a gauge invariant expression for the back-reaction of density inhomogeneities on the global expansion of perturbed FLRW spacetimes, in terms of observable quantities, and we calculate the effect quantitatively. Since we do not adopt a comoving gauge, our result incorporates the back-reaction on the metric due to scalar velocity and vorticity perturbations. The results are compared with the results by other authors in this field.

## 1 Introduction

An essential difficulty which occurs when dealing with realistic cosmological models, is related to the fact that although the universe seems to be very close to FLRW at length scales of the order of the Hubble radius, the metric and matter content of the universe appears to be highly inhomogeneous at smaller scales. Since the realistic universe, with all its details at length scales small compared to the Hubble radius, is too complicated to handle in most calculations, it seems desirable to extract those physical quantities which describe the large scale structure of the universe. However, when one tries to define an averaging operation for metrics, a number of difficulties occur. One of these difficulties is related to the fact that the Einstein equations are inherently non-linear, which makes it a nontrivial question to see how the Einstein equations constrain the dynamics of an averaged metric. Another fundamental problem which occurs when one tries to average metrics, is related to the fact that there is generally no direct physical significance in an averaged metric. Although this problem is usually ignored in the literature on averaging, it needs to be addressed before one can extend the discussion on averaging beyond an intuitive level of understanding. The usual approach to averaging (see e.g., [1] - [5]) seems to be that one defines an averaging method, which is chosen on the basis of mathematical elegance or an intuitive notion of smoothness, and then one

defines averaged physical quantities by means of the averaging operation which one has chosen. The objection against this approach is that if one calculates e.g. the averaged expansion of a perturbed FLRW universe, one can obtain virtually any result, by choosing an averaging operation which yields this specific result. In section 2 we explicify this problem, and we derive a generic linearized averaging operation for *metrics* which satisfies the condition that unperturbed FLRW is a *stable fixed point* of the averaging operation. It is shown that this generic linearized averaging operation for metrics can be expressed in terms of the spatial average of the perturbation of the spatial volume and  $g_{00}$  in coordinates which are synchronous in the background. In section 3 we discuss the gauge problem and the choice of the background spacetime. The averaged constraint equations are explicitly evaluated in section 4, and in subsection 4.1 we derive an expression for the correction to averaged expansion due to density perturbations, in terms of the power spectrum of the matter. In subsections 4.2 and 4.3 we discuss the back-reaction on the metric due to matter velocity perturbations, and we show that vorticity perturbations may be important in the large wavelength limit. In section 4.4 we calculate the different corrections to the averaged expansion quantitatively by means of the observational data, and we compare our results with the results derived in previous works. In this paper, we adopt the convention that greek indices run from 0 to 3, while latin indices run from 1 to 3. The metric signature is  $(-+++)$ , and the velocity of light,  $c$ , is set equal to one.

## 2 The spatial average

In this section we will consider the generic linearized averaging operation for metrics, for which unperturbed FLRW is a stable fixed point, and we show that this averaging operation has a universal limit when applied iteratively to perturbed FLRW spacetimes. We determine this limit explicitly, and by using the symmetry of the background, we show that the averaging of the 10 components of the metric perturbation  $\delta g_{\rho\sigma}$ , can be expressed in terms of the spatial average of  $g_{00}$  and  $\sqrt{g^{(3)}}$  in coordinates which are synchronous in the background. From now on the background FLRW spacetime will be called  $\bar{\mathbf{S}}$ , while the inhomogeneous spacetime is called  $\mathbf{S}$ . Furthermore, we assume that  $\bar{\mathbf{S}}$  is coordinatized such that  $t$  represents the time coordinate which labels the hypersurfaces of homogeneity  $\bar{\Sigma}$  in  $\bar{\mathbf{S}}$ , and  $\bar{\Sigma}$  is coordinatized by  $x^i$  where  $i \in \{1, 2, 3\}$ . We call a metric  $g_{\mu\nu}$  or a metric perturbation  $\delta g_{\mu\nu}$  spatially homogeneous and isotropic when there exists at least one coordinate system in which the components of  $g_{\mu\nu}$  or  $\delta g_{\mu\nu}$  are spatially homogeneous and invariant under spatial rotations.

Let us consider the most general averaging operation  $\hat{A}$ , which is a functional  $\mathcal{F}$  of metric perturbations  $\delta g_{\mu\nu}$  about some background solution  $\bar{\mathbf{S}}$ ,

$$\hat{A}\delta g_{\mu\nu}(x) = \mathcal{F}_{\mu\nu}(\delta g_{\rho\sigma}(x)). \quad (1)$$

We will *require* the condition that unperturbed FLRW, in a gauge where the metric perturbation  $\delta g_{\mu\nu}$  is spatially homogeneous and isotropic, is a stable

fixed point of the averaging operation  $\hat{A}$ . This condition states that the averaging operation increases the spatial symmetry of the spacetime on which it works, assuming that this spacetime is sufficiently ‘close’ to FLRW, and it defines what we mean by averaging in this paper.

It follows directly from this assumption that

$$\mathcal{F}_{\mu\nu}(0) = 0, \quad (2)$$

for all  $x$ , since a nonzero value at the right-hand side of equation (2) implies that unperturbed FLRW with the same geometry as  $\bar{\mathbf{S}}$ , in a gauge where  $\delta g_{\mu\nu} = 0$  for all  $x$ , is not a fixed point of  $\hat{A}$ , which contradicts our assumption.

The *linear approximation* to the averaging operation (1), is given by

$$\hat{A}^{(1)}\delta g_{\mu\nu}(x) = \int_{\bar{\mathbf{S}}} d^4x' f_{\mu\nu}^{\rho\sigma}(x, x') \delta g_{\rho\sigma}(x'), \quad (3)$$

where the bi-tensor density  $f_{\mu\nu}^{\rho\sigma}(x, x')$  is defined as the functional derivative of  $\mathcal{F}_{\mu\nu}$  with respect to  $\delta g_{\rho\sigma}$ , evaluated at the point with coordinates  $x'$  in the background, i.e.,

$$f_{\mu\nu}^{\rho\sigma}(x, x') := \left. \frac{\partial \mathcal{F}_{\mu\nu}(g, x)}{\partial g_{\rho\sigma}(p)} \right|_{\delta g_{\rho\sigma}=0, p=x'}, \quad (4)$$

and we used condition (2) which states that the zeroth order contribution in the expansion of  $\hat{A}$  vanishes.

The condition that unperturbed FLRW is a *stable* fixed point of the averaging operation  $\hat{A}$  implies that the limit

$$\hat{A}^\infty \delta g_{\mu\nu} := \lim_{n \rightarrow \infty} \hat{A}^{(1)n} \delta g_{\mu\nu} \quad (5)$$

exists, and the quantity  $\hat{A}^\infty \delta g_{\mu\nu}$  must be spatially homogeneous and isotropic (we used the notation  $\hat{A}^{(1)n}$  to denote the  $n$ -times repeated operation of  $\hat{A}^{(1)}$ ).

Note that the averaging operation  $\hat{A}$  has two aspects; first it changes the geometry of the spacetime on which it works, and second it specifies a correspondence between points in the spacetime  $\mathbf{S}$ , the averaged spacetime  $\hat{A}\mathbf{S}$ , and the background  $\bar{\mathbf{S}}$ .

When one only requires that unperturbed FLRW is a stable fixed point of  $\hat{A}$ , one constrains the way in which  $\hat{A}$  changes the geometry of the spacetime on which it works, but one does not constrain the correspondence between points in the spacetimes  $\mathbf{S}$ ,  $\hat{A}\mathbf{S}$  and  $\bar{\mathbf{S}}$ . We constrain this freedom by imposing the stronger requirement that unperturbed FLRW, in a gauge where the metric perturbation  $\delta g_{\mu\nu}$  is spatially homogeneous and isotropic, is a stable fixed point of  $\hat{A}$ . This condition enforces that  $\hat{A}$  does not generate ‘pure gauge’ perturbations when operating on unperturbed FLRW.

Starting from equation (5), and using the symmetries of the background spacetime  $\bar{\mathbf{S}}$ , it is shown in appendix B that the averaging operation  $\hat{A}^\infty$  can be defined in terms of a spatial averaging operation which is universal, i.e.,

$$\hat{A}^\infty \delta g_{\mu\nu}(t, x^i) = \langle \delta g_{\mu\nu} \rangle(t), \quad (6)$$

where

$$\begin{aligned} \langle \delta g_{\mu\nu} \rangle(t) = & \int_{\bar{\Sigma}(t)} d^3x' \alpha \sqrt{g^{(3)}} \\ & \times \left[ \bar{n}^\rho(x') \bar{n}^\sigma(x') \bar{n}_\mu(x) \bar{n}_\nu(x) + \frac{1}{3} \bar{h}^{\rho\sigma}(x') \bar{h}_{\mu\nu}(x) \right] \delta g_{\rho\sigma}(x'), \end{aligned} \quad (7)$$

where  $\bar{n}^\rho$  denotes the future directed unit vector normal to  $\bar{\Sigma}$ , and  $\bar{h}^{\rho\sigma} := g^{B\rho\sigma} + \bar{n}^\rho \bar{n}^\sigma$  is the projection operator on  $\bar{\Sigma}$ , and  $\alpha$  denotes the distribution which is constant on  $\Sigma$ , and for which the integral over  $\Sigma$  equals one. Note that  $\bar{n}^\rho \bar{n}^\sigma \delta g_{\rho\sigma}$  equals the perturbation of  $g_{00}$  in coordinates which are synchronous in the background (i.e., coordinates for which  $g_{\mu 0}^B = -\delta_\mu^0$ ), while  $\bar{h}^{\rho\sigma} \delta g_{\rho\sigma}$  equals the perturbation of the spatial volume element on  $\Sigma$ , to first order. It follows from this observation that the linearized averaging operation for metrics (7), is effectively a spatial averaging operation for scalars, applied to  $\delta g_{00}$  and  $\delta g^i_i$  in coordinates which are synchronous in the background.

An explicit realization of the spatial averaging operation for a scalar  $q(x)$ , in the case where  $\bar{\Sigma}$  is open, is given by

$$\langle q(x) \rangle = \lim_{\ell \rightarrow \infty} \langle q(x) \rangle(\ell) \quad (8)$$

$$:= \lim_{\ell \rightarrow \infty} N^{-1}(x, \ell) \int_{\bar{\Sigma}} d^3x' \sqrt{g^{(3)}} q(x') \theta(\ell - \Delta s(x, x')),$$

where  $N(x, \ell) := \int_{\bar{\Sigma}} d^3x' \sqrt{g^{(3)}} \theta(\ell - \Delta s(x, x'))$ , and  $\Delta s(x, x')$  is a distance measure between points  $x$  and  $x'$ ,  $\ell$  is a parameter with the dimension of length, and  $\theta(x) = 1(0)$  for  $x \geq 0(x < 0)$ . In the case where  $\Sigma$  is closed,  $\langle q \rangle$  is defined analogously to expression (8), with  $N(x, \ell) = \text{volume}(\Sigma)$ .

It is shown in appendix A that the spatial average of a scalar function is invariant under *spatial* gauge transformations, to arbitrary order in the expansion parameter of the gauge transformation.

Notice that the spatial average (8) is only well defined when we make the assumption that perturbations  $q(x)$  are sufficiently small, such that the limit  $\ell \rightarrow \infty$  in equation (8) exists. It should be stressed that this assumption is nontrivial, and it is not automatically satisfied in general cosmological situations, where perturbations are not necessarily bounded in amplitude and length scale. Indeed, since the observable part of our universe is restricted to our past light cone, there is no observational basis for the assumption that our universe is 'close' to FLRW at arbitrary large length scales. The usual way to deal with this situation is that one adopts *a priori* philosophical assumptions, such as the Copernican principle, to choose between different models which satisfy the observational data (see e.g., [8]). Throughout this paper, we will adopt a version of the Copernican principle by assuming that perturbations are small enough such that the limit  $\ell \rightarrow \infty$  in equation (8) exists.

### 3 The gauge problem

As is pointed out by Futamase in [1], the observed matter density contrasts are of the order of unity at dimensionless length scales  $\kappa$  of the order of  $10^{-2}$ ,

where  $\kappa$  denotes the fraction of typical size of the density fluctuation and the Hubble radius  $r_H := c/H_0$ . A rough estimation of the order of magnitude of the associated Newtonian gravitational potential, which we call  $\epsilon$  from now on, can be obtained by using the Poisson equation. For density contrasts of the order of unity we find  $\epsilon \sim \kappa^2$ , which implies a Newtonian potential  $\phi$  of the order of  $10^{-4}$ , suggesting that a perturbative approach might be adequate. At length scales of the order of the Hubble radius, the observable part of the universe appears to be highly homogeneous and isotropic, which motivates our choice for the FLRW metric as a background metric.

Let us first briefly discuss some details concerning the spherical harmonic decomposition of perturbations about a background FLRW spacetime.

The FLRW background metric can be written in the form,

$$ds^2 = g_{\mu\nu}^B dx^\mu dx^\nu = a^2(\bar{t})(-d\bar{t}^2 + \eta_{ij} dx^i dx^j), \quad (9)$$

where  $\eta_{ij}$  is the metric tensor for a homogeneous and isotropic three-space with curvature  $\mathbf{k}$ , and  $\bar{t}$  is a *conformally scaled* time parameter. We define the metric perturbation  $h_{\mu\nu}$  by,

$$g_{\mu\nu} = g_{\mu\nu}^B + h_{\mu\nu} \quad , \quad g^{\mu\nu} = g^{B\mu\nu} - h^{\mu\nu}, \quad (10)$$

and since  $g^{\mu\rho} g_{\mu\nu} = \delta_\nu^\rho$ , we have  $h^{\mu\nu} = g^{B\nu\rho} g^{B\mu\sigma} h_{\rho\sigma}$ , and  $h^\mu_\nu = g^{B\mu\rho} h_{\rho\nu}$ , to first order.

Copying Bardeen's notation in [6], we define scalar, vector and tensor spherical harmonics  $Q_n^{(0)}$ ,  $Q_{ni}^{(1)}$  and  $Q_{nij}^{(2)}$ , respectively, which satisfy the Helmholtz equations  $Q_n^{(p)|q} + k_n^2 Q_n^{(p)} = 0$ , where  $p \in \{0, 1, 2\}$  and  $|$  denotes the covariant derivative with respect to  $g_{ij}^B$ . The vector harmonics  $Q_i^{(1)}$  are divergenceless, while the tensor harmonics  $Q_{ij}^{(2)}$  are divergenceless, symmetric, and traceless. We define traceless symmetric scalar harmonics  $Q_{nij}^{(0)}$  by

$$Q_{nij}^{(0)} := k_n^{-2} Q_{n|ij}^{(0)} + \frac{1}{3} g_{ij}^B Q_n^{(0)} \quad \text{and} \quad Q_n^{(0)|ij} - (k_n^2 - 3\mathbf{k}) Q_n^{(0)} = 0, \quad (11)$$

and traceless symmetric vector harmonics  $Q_{nij}^{(1)}$  are defined by

$$Q_{nij}^{(1)} := -\frac{1}{2k_n} (Q_{ni|j}^{(1)} + Q_{nj|i}^{(1)}) \quad \text{and} \quad Q_{nij}^{(1)|i} - (k_n^2 - 2\mathbf{k}) Q_{nj}^{(1)} = 0. \quad (12)$$

The spherical harmonics are labeled by the parameter  $n \in 0, \mathbf{Z}^+$  ( $\vec{k} \in \mathbf{R}^3$ ) in the case where  $\bar{\Sigma}$  is closed (open). It is useful to define the hypersurface integration operation for scalars  $q(x)$  by

$$\langle\langle q \rangle\rangle := \langle q (g^{(3)B}/g^{(3)})^{\frac{1}{2}} \rangle, \quad (13)$$

which differs from the spatial average (8) by the volume element which is evaluated in the background. As we show in appendix 2, the spatial average (8) of a physical quantity is invariant under spatial gauge transformations, while the hypersurface integral (13) is generally gauge dependent at second and higher order in the expansion parameter of the gauge transformation.

The spherical harmonics  $Q_n^{(0)}$ ,  $Q_n^{(1)}$  and  $Q_n^{(2)}$  are orthogonal with respect to the hypersurface integration operation, i.e.,

$$\langle\langle Q_n^{(0)} Q_{n'}^{(0)} \rangle\rangle = \langle\langle Q_n^{(1)} Q_{n'}^{(1)i} \rangle\rangle = \langle\langle Q_n^{(2)} Q_{n'}^{(2)ij} \rangle\rangle = \delta_{nn'}, \quad (14)$$

and

$$\frac{3}{2} \langle\langle Q_{nij}^{(0)} Q_{n'}^{(0)ij} \rangle\rangle = 2 \langle\langle Q_{nij}^{(1)} Q_{n'}^{(1)ij} \rangle\rangle = \delta_{nn'}. \quad (15)$$

Notice that the spherical harmonics are only to zeroth order orthogonal with respect to the spatial averaging operation (8) due to a generally nonvanishing first order term which arises from the expansion of the volume element  $\sqrt{g^{(3)}} = \sqrt{g^B}(1 + h + O(h^2))$ .

The most general representation of a symmetric  $4 \times 4$  tensor  $h_{\mu\nu}$  in terms of the complete basis of spherical harmonics is given by

$$\begin{aligned} h_{00} &= -2a^2 \sum_n A_n Q_n^{(0)} \\ h_{0i} &= -a^2 \sum_n [B_n^{(0)} Q_{ni}^{(0)} + B_n^{(1)} Q_{ni}^{(1)}] \\ h_{ij} &= 2a^2 \sum_n [H_{Ln}^{(0)} g_{ij}^B Q_n^{(0)} + H_{Tn}^{(0)} Q_{nij}^{(0)} + H_{Tn}^{(1)} Q_{nij}^{(1)} + H_{Tn}^{(2)} Q_{nij}^{(2)}], \end{aligned} \quad (16)$$

where the coefficients  $A_n, B_n^{(0)}, B_n^{(1)}, H_{Tn}^{(0)}, H_{Tn}^{(1)}$  and  $H_{Tn}^{(2)}$  are generally dependent on the conformal time parameter  $\bar{t}$ . Let  $u^\mu$  be the four-velocity associated with the frame in which the energy flux of the matter vanishes, then the three-velocity  $u^i/u^0$  associated with  $u^\mu$ , can be expanded as

$$u^i/u^0 = \sum_n [v_n^{(0)} Q_{ni}^{(0)} + v_n^{(1)} Q_{ni}^{(1)}], \quad (17)$$

where  $Q_{ni}^{(0)} := -k_n^{-1} Q_{n|i}^{(0)}$ , and  $u^0 = 1/a(\bar{t})$  to first order, due to the normalization  $u_\mu u^\mu = -1$ .

A gauge transformation is defined as a change in the correspondence between points  $p$  in  $S$ , and points  $\bar{p}$  in  $\bar{S}$ . The most general first order gauge transformation is the result of the coordinate transformation

$$\tilde{t} = t + \sum_n T_n Q_n^{(0)}(x^\mu), \quad (18)$$

$$\tilde{x}^i = x^i + \sum_n (L_n^{(0)} Q_n^{(0)i}(x^\mu) + L_n^{(1)} Q_n^{(1)i}(x^\mu)), \quad (19)$$

in  $S$ , while the coordinates in  $\bar{S}$  are fixed, and the correspondence between points with the same coordinates in  $S$  and in  $\bar{S}$  is kept fixed. The coefficients  $T_n$  and  $L_n$  in expression (18) and (19) are arbitrary functions of the conformal time coordinate  $\bar{t}$ . Notice that  $T_n$  generates a change in the correspondence of the time coordinates in  $S$  and  $\bar{S}$ , while  $L_n$  generates a change in the correspondence between the spatial hypersurface coordinates on  $\Sigma$  and  $\bar{\Sigma}$ . The changes in the

amplitudes of the metric tensor are calculated in the case of scalar perturbations [6],

$$\tilde{A}_n = A_n - \dot{T}_n - \frac{\dot{a}}{a} T_n, \quad (20)$$

$$\tilde{B}_n^{(0)} = B_n^{(0)} + \dot{L}_n^{(0)} + k_n T_n, \quad (21)$$

$$\tilde{H}_{Ln}^{(0)} = H_{Ln}^{(0)} - (k_n/3) L_n^{(0)} - \frac{\dot{a}}{a} T_n, \quad (22)$$

$$\tilde{H}_{Tn}^{(0)} = H_{Tn}^{(0)} + k_n L_n^{(0)}, \quad (23)$$

where a dot denotes conformal time differentiation. For vector perturbations we find,

$$\tilde{B}_n^{(1)} = B_n^{(1)} + \dot{L}_n^{(1)}, \quad \tilde{H}_{Tn}^{(1)} = H_{Tn}^{(1)} + k_n L_n^{(1)}, \quad (24)$$

while for tensor perturbations the trivial relation  $\tilde{H}_{Tn}^{(2)} = H_{Tn}^{(2)}$  holds for all  $n$ . The matter velocity perturbation coefficients  $v_n^{(0)}$  and  $v_n^{(1)}$ , with respect to the coordinate frame, transform as,

$$\tilde{v}_n^{(i)} = v_n^{(i)} + \dot{L}_n^{(i)}, \quad (25)$$

where  $i \in \{1, 2\}$ . Apart from the gauge freedom which is related to the mapping between points in  $S$  and  $\tilde{S}$ , there is a gauge freedom related to the choice of the background scale factor  $a(\bar{t})$ . A first order change in the choice of the background scale factor,

$$\tilde{a}(\bar{t}) = a(\bar{t}) + D(\bar{t}) \quad (26)$$

affects the spatially homogeneous mode of the trace part of the spatial metric by a change,

$$\tilde{H}_{L0}^{(0)} = H_{L0}^{(0)} + \frac{D}{a}, \quad (27)$$

where  $H_{L0}^{(0)}$  is the coefficient which multiplies the spatially homogeneous trace mode in the expansion of the metric (16), and

$$H_{L0}^{(0)} = \frac{1}{3} \langle \langle \frac{\sqrt{g^{(3)}}}{\sqrt{g^B}} - 1 \rangle \rangle, \quad (28)$$

to first order.

The approach in this paper will be based on a specification of the temporal part of the gauge (i.e., the correspondence between the time coordinates  $t$  in  $S$  and  $\bar{t}$  in  $\tilde{S}$ ), while maintaining covariance with respect to spatial gauge transformations (i.e., the correspondence between the spatial coordinates  $x^i$  in  $S$  and  $\bar{x}^i$  in  $\tilde{S}$ ).

We stress that a fully gauge covariant approach is preferred to an approach which is based on a (partially) fixed gauge, since explicitly gauge dependent results generally point out nonphysical features of the calculation, including calculational mistakes. However, the intricateness of a fully gauge covariant calculation at second order makes such a calculation cumbersome (see e.g. [10]), and we will therefore follow an approach where the temporal part of the gauge is fixed, while maintaining spatial gauge covariance.

The temporal inhomogeneous part of the gauge is specified by imposing conditions on the coefficients in the expansion of the metric (16). The extrinsic curvature tensor of the constant- $t$  hypersurfaces in  $S$  is given by,

$$K_j^i = \frac{1}{a} \sum_n \left[ \frac{\dot{a}}{a} + \left( \dot{H}_{Ln}^{(0)} - \frac{\dot{a}}{a} A_n + \frac{k}{3} B_n^{(0)} \right) Q_n^{(0)} \right] \delta_j^i + \left[ \dot{H}_{Tn}^{(0)} - k B_n^{(0)} \right] Q_{nj}^{(0)i} + \left[ \dot{H}_{Tn}^{(1)} - k B_n^{(1)} \right] Q_{nj}^{(1)i} + \dot{H}_{Tn}^{(2)} Q_{nj}^{(2)i}. \quad (29)$$

By requiring that the coefficients  $A_n, B_n^{(0)}$  and  $H_{Ln}^{(0)}$  in the expansion of the metric (16) satisfy the condition

$$\dot{H}_{Ln}^{(0)} - \frac{\dot{a}}{a} A_n + \frac{k}{3} B_n^{(0)} = 0, \quad (30)$$

for all  $n$ , we specify a gauge in which the hypersurfaces of constant time  $t$  in  $S$  have spatially constant volume expansion  $K = 3\dot{a}/a^2$ , as is clear by contracting expression (29). Condition (30) specifies more or less uniquely a collection of spatial hypersurfaces in  $S$  (see [11]), but uniqueness is not required in the calculation which follows, since, as we will show in the following, our result for the average expansion of an inhomogeneous universe does not in relevant order depend on the choice of the inhomogeneous temporal part of the gauge.

Note that condition (30) does not constrain the choice of the time coordinate in  $S$ , and the correspondence between the time coordinates  $t$  in  $S$  and  $\bar{t}$  in  $\bar{S}$ . We specify the time parameter  $t$  in  $S$ , up to the freedom of adding a constant, by imposing the requirement that the homogeneous component of  $A$  vanishes, i.e.,

$$A_0 = 0, \quad (31)$$

for all times  $\bar{t}$ .

The choice of gauge (31) implies that the background time interval coincides with the averaged proper time interval in  $S$ , as measured by observers which are comoving with the spatial coordinates.

The gauge condition (31) can always be satisfied by performing a first order homogeneous gauge transformation. In order to clarify this statement, let us consider how equation (31) is affected by a homogeneous temporal gauge transformation. According to expression (20), a homogeneous temporal gauge transformation with  $T = T_0$ , induces a first order change in the metric perturbation coefficient  $A_0$ ,

$$\tilde{A}_0 = A_0 - \dot{T}_0 - \frac{\dot{a}}{a} T_0. \quad (32)$$

The gauge condition (31) is satisfied by performing a gauge transformation of the form (20), where

$$T_0 = \frac{c}{a(\bar{t})} + \frac{1}{a(\bar{t})} \int^{\bar{t}} d\tau a(\tau) A_0(\tau), \quad (33)$$

and  $c$  is a constant of integration. The gauge condition (31) therefore determines the homogeneous temporal part of the gauge, up to a constant of integration  $c$ . According to the transformation law (22), the constant of integration



$c$  in expression (33) affects the spatially homogeneous trace part of the spatial metric. This gauge freedom can be fixed by requiring that the homogeneous trace perturbation of the spatial metric vanishes, i.e.,

$$H_{L0}^{(0)} = 0, \quad (34)$$

for one time  $\bar{t}_c$  and for a fixed choice of the background scale factor  $a(\bar{t})$  at  $\bar{t} = \bar{t}_c$ .

Although we have completely specified the homogeneous temporal part of the gauge by imposing the gauge conditions (31) and (34), the homogeneous trace perturbation of the spatial metric may still differ from zero for times  $\bar{t} \neq \bar{t}_c$ . These perturbations are related to the freedom of choosing the background scale factor  $a(\bar{t})$  for times  $\bar{t} \neq \bar{t}_c$ , as is clear from equation (26). When we require that condition (34) holds at *all* times  $\bar{t}$ , then it follows from equation (26) and (27) that the choice of the background scale factor  $a(\bar{t})$  is fixed for all times  $\bar{t} \in \mathbf{R}$ .

Recall that in section (2) we derived the generic linearized averaging operation for which unperturbed FLRW is a stable fixed point. It was shown that this linearized averaging operation which works on the ten components of the metric tensor, reduces to evaluating the spatial average of  $\delta g_i^i$  and  $\delta g_{00}$ . By imposing the gauge conditions (31) and (34), we specified a choice of background geometry by requiring that the spatial averages of  $\delta g_i^i$  and  $\delta g_{00}$  both vanish. For this choice of gauge, the averaged spacetime equals the background spacetime, and the averaging problem reduces to solving the averaged constraint equations for the background scale factor  $a(\bar{t})$ .

An explicit expression for the background scale factor  $a(\bar{t})$  in the gauge fixed by condition (34) is obtained by substituting the expression for the background metric (9) and expression (16) for the perturbed metric, into expression (28). To first order we find,

$$a^3(\bar{t}) = \langle \langle \frac{\sqrt{g^{(3)}}}{\sqrt{\eta}} \rangle \rangle, \quad (35)$$

where  $g^{(3)} = \det(g_{ij})$  and  $\eta = \det(\eta_{ij})$ .

Recall that condition (30) fixes the inhomogeneous temporal part of the gauge, and the collection of spatial hypersurfaces on which the spatial average is evaluated. Since physical results must be gauge invariant to relevant order, one may question whether the freedom of choosing a family of hypersurfaces affects the result for the scale factor (35). It follows from the orthonormality relation (14) and the transformation property (22) that the background scale factor (35) is invariant to first order under inhomogeneous temporal gauge transformations. However, at order  $\epsilon^2$  inhomogeneous metric perturbations do contribute to the background scale factor (35), and the gauge invariance of the scale factor  $a(\bar{t})$  therefore breaks down at order  $\epsilon^2$ . Consistent with this limitation we will neglect terms of order  $\epsilon^2$  in our calculation, while retaining terms of order  $\epsilon$  and  $\epsilon^2/\kappa^2$ .

Summarizing the content of this subsection, we completely specified the temporal and the spatially homogeneous part of the gauge, and the choice of the background, by imposing the gauge conditions (30) (31) and (34) on the metric coefficients  $A_n, B_n^{(0)}$  and  $H_{Ln}^{(0)}$ .

## 4 Averaging the constraint equations

The classical constraint equations on a hypersurface  $\Sigma$  are given by

$$R^{(3)} + K^2 - K_{ij}K^{ij} = 16\pi G\rho + 2\Lambda, \quad (36)$$

$$K_{i;j}^j - K_{;i} = 8\pi GJ_i, \quad (37)$$

where  $;$  denotes the covariant derivative with respect to  $g_{ij}$ , and  $R^{(3)}$  is the Ricci scalar associated with the induced metric  $g_{ij}$ , and

$$\rho = T^{\mu\nu}n_\mu n_\nu, \quad J_i = -T^{\mu\nu}h_{i\mu}n_\nu, \quad (38)$$

where  $n_\mu$  denotes the future directed unit vector normal to  $\Sigma$ , and  $h_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$ .

In the constant- $K$  gauge, defined by conditions (30), (31) and (34), the constraint equation (36) takes the form

$$\frac{\dot{a}^2}{a^4} = \frac{8\pi}{3}G\rho - \frac{1}{6}R^{(3)} + \frac{1}{6}\hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\Lambda, \quad (39)$$

where  $\hat{K}^{ij} := K^{ij} - \frac{1}{3}g^{ij}K$  is the traceless part of the extrinsic curvature tensor.

In principle one could solve the constraint equation (39) for the time dependence of the scale factor  $a(t)$ , while taking into account all linear and higher order contributions to the right-hand side of equation (39). However, this approach is unnecessarily complicated, since all terms which do not have constant values on  $\Sigma$  must cancel on the right-hand side of equation (39), since the left-hand side of equation (39) is constant on  $\Sigma$ . For the sake of *calculational* convenience, we will take the spatial average at the right-hand side of the constraint equation (39), without changing any physical aspects of the constraint equation;

$$\frac{\dot{a}^2}{a^4} = \frac{8\pi}{3}G\langle\rho\rangle - \frac{1}{6}\langle R^{(3)}\rangle + \frac{1}{6}\langle\hat{K}_{ij}\hat{K}^{ij}\rangle + \frac{1}{3}\Lambda. \quad (40)$$

In order to solve equation (40) for the scale factor  $a(t)$ , we need to evaluate the spatial average of the 3-curvature  $R^{(3)}$ , and the energy density  $\rho$ , and the square of the traceless part of the extrinsic curvature tensor  $\hat{K}_{ij}\hat{K}^{ij}$ . We will calculate these quantities in the following three subsections.

### 4.1 The averaged spatial curvature

The spatial curvature perturbation  $\delta R^{(3)}$  can be expanded in terms of the 3-metric perturbation  $h_{ij}$  (see e.g. [9]),

$$R^{(3)} = \frac{6\mathbf{k}}{a^2} + \delta R^{(3)}, \quad (41)$$

where

$$\delta R^{(3)} = \delta R + \delta^2 R + O(h^3), \quad (42)$$

and

$$\delta R = h_{|k}^{|k} - h_{i|k}^{k|i}, \quad (43)$$

$$\delta^2 R = -\frac{1}{4}h^{ij}h_{ij|q}^{|q} + \frac{1}{2}h^{ij}h_{qi|j}^{|q} - \frac{1}{4}hh_{|l}^{|l} + \text{td.}, \quad (44)$$

where td. stands for terms which are total derivatives. Let us now evaluate the contributions to the averaged curvature perturbation  $\langle R^{(3)} \rangle$  for scalar, vector, and tensor modes in the expansion of  $h_{ij}$ .

#### 4.1.1 Scalar perturbations

It follows from expression (42) that the lowest order contribution to the spatial curvature perturbation is given by  $g^{Bij}\delta R_{ij}$ , which is order  $\epsilon/\kappa^2$ , but the spatial average of this contribution vanishes to order  $\epsilon/\kappa^2$ , due to the orthogonality relations (14). The linear curvature perturbation  $g^{Bij}\delta R_{ij}$  does however contribute to the averaged 3-curvature perturbation by a term of order  $\epsilon^2/\kappa^2$ , i.e.,

$$\langle \delta R \rangle = \frac{12}{a^2} \sum_n (k_n^2 - 3\mathbf{k}) H_{Ln}^{(0)} (H_{Ln}^{(0)} + \frac{1}{3} H_{Tn}^{(0)}) + O(\epsilon^3/\kappa^2), \quad (45)$$

where we made use of the expansion of the volume element  $\sqrt{g^{(3)}} = \sqrt{g^B}(1 + h + O(h^2))$ , and the definition of the spatial average (8).

The quadratic term  $\delta^2 R$  in the expansion of the 3-curvature perturbation (42) contributes to the averaged 3-curvature perturbation by a term

$$\langle \delta^2 R \rangle_\infty^{(0)} = -\frac{1}{a^2} \sum_n (k_n^2 - 3\mathbf{k}) (10H_{Ln}^{(0)2} - \frac{2}{9}H_{Tn}^{(0)2} + \frac{8}{3}H_{Tn}^{(0)}H_{Ln}^{(0)}) + O(\epsilon^3/\kappa^2), \quad (46)$$

where we used the computer algebra package MAPLE to derive this expression. Combining expression (45) and (46), we find an expression for the scalar contribution to the spatial curvature perturbation,

$$\langle \delta R^{(3)} \rangle^{(0)} = \frac{2}{a^2} \sum_n (k_n^2 - 3\mathbf{k}) \phi_{hn}^2 + O(\epsilon^3/\kappa^2), \quad (47)$$

where

$$\phi_{hn} := H_{Ln}^{(0)} + \frac{1}{3}H_{Tn}^{(0)} \quad (48)$$

is the gauge invariant amplitude which measures the distortion of the intrinsic geometry of the constant- $K$  hypersurfaces. Using the expansion (42) for the spatial curvature perturbation, and the definition (48) of  $\phi_{hn}$ , one finds that  $\phi_{hn}$  is related to the *first order* spatial curvature perturbation by,

$$\delta R = \frac{4}{a^2} \sum_n (k_n^2 - 3\mathbf{k}) \phi_{hn} Q_n^{(0)}. \quad (49)$$

By substituting expression (49) into the constraint equation (36), we obtain a simple expression for  $\phi_{hn}$  in terms of the first order energy perturbation,

$$\phi_{hn} = \frac{4\pi a^2}{(k_n^2 - 3\mathbf{k})} G\bar{\rho} \epsilon_{hn} + O(\epsilon^2/\kappa^2), \quad (50)$$

for all  $n$ , where  $\epsilon_{hn}$  is defined as the density contrast in the constant- $K$  gauge,

$$\epsilon_{hn} = \delta\rho(k_n)/\bar{\rho}, \quad (51)$$

and  $\bar{\rho}$  denotes the background energy density.

We would like to express the scalar contribution to the averaged curvature perturbation in terms of observable quantities. Since the averaged curvature perturbation (47) is quadratic in  $\phi_{hn}$ , we may use the constraint equation (50) to *first order* to determine  $\phi_{hn}$  in terms of the fractional energy perturbation  $\epsilon_{hn}$ . We obtain,

$$\langle \delta R^{(3)} \rangle^{(0)} = 32(\pi a G \bar{\rho})^2 \sum_n \frac{\epsilon_h^2(q_n)}{(k_n^2 - 3\mathbf{k})} + O(\epsilon^3/\kappa^2), \quad (52)$$

where the sum (or integral when  $\Sigma$  is open) is taken over all possible values  $n$ . Expression (52) takes an especially simple form when expressed in terms of the power spectrum  $P(k)$ , which allows the representation

$$P_h(k) = \sum_n \frac{\epsilon_h^2(q_n)}{4\pi q_n^2} \delta(k - |q_n|), \quad (53)$$

where the subscript  $h$  refers to the constant- $K$  gauge (see e.g. [12] or [13] for more on power spectra). Combining expressions (52) and (53) yields,

$$\langle \delta R^{(3)} \rangle^{(0)} = 32(\pi^2 G \bar{\rho})^2 J_2 + O(\epsilon^2, \epsilon^3/\kappa^2), \quad (54)$$

where

$$J_2 := 4\pi a^2 \int_0^\infty dk P_h(k) \quad (55)$$

is an observable quantity known as the second moment of the power spectrum, and by absorbing a factor  $a^2$  in the definition of  $J_2$  we restored physical units of length square.

#### 4.1.2 Vector perturbations

Using the definition of the vector harmonics (12), and the orthogonality relations (14), we find that vector perturbations do not contribute to the spatial curvature perturbation (42). This result may be expected, since it follows from expression (24) that one can always choose a gauge in which there are no vector perturbations of the spatial metric, and the vector contribution to the averaged spatial curvature perturbation (42) must therefore vanish in any gauge, due to gauge invariance of the averaged spatial curvature perturbation.

#### 4.1.3 Tensor perturbations

Using equations (14), and expression (42) for the second order expansion of the spatial curvature, it follows immediately that,

$$\langle \delta R^{(3)} \rangle^{(2)} = \frac{1}{a^2} \sum_n k_n^2 H_{Tn}^{(2)2}, \quad (56)$$

while the tensor contribution to the term  $\langle \hat{K}_{ij} \hat{K}^{ij} \rangle$  in the averaged constraint equation (39) follows immediately from the expression for the extrinsic curvature tensor (29). Although the tensor contribution to the averaged constraint equations is easily calculated in terms of the coefficients  $H_{Tn}^{(2)}$ , the magnitude of this term has not yet been determined quantitatively by the observation of gravitational waves.

## 4.2 Averaged energy density

In this subsection we will calculate the averaged energy density  $\langle \rho \rangle$ . In order to calculate the lowest order nontrivial contribution to the averaged energy perturbation, we will adopt the assumption in this subsection that the matter in the universe at late times after decoupling can be effectively described by the energy momentum tensor density for a pressureless perfect fluid, i.e.,

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu, \quad (57)$$

where  $u^\mu$  is the four-velocity of the fluid, and  $\rho_0$  is the energy density in the rest-frame of the fluid. The equations of motion for the fluid read,

$$\nabla_\mu T^{\mu\nu} = 0, \quad (58)$$

which implies

$$\partial_\mu (\sqrt{-g} \rho_0 u^\mu) = 0, \quad (59)$$

where we used that  $\nabla_\mu = (\sqrt{-g})^{-\frac{1}{2}} \partial_\mu \sqrt{-g}$ , and  $u^\nu \nabla_\nu u^\mu = 0$  for a pressureless fluid. By using the spatial gauge freedom (19) we may set  $B_n^{(0)} = B_n^{(1)} = 0$ , such that  $\sqrt{-g} = \sqrt{-g_{00}} \sqrt{g^{(3)}}$ , and the equation of motion (59) takes the form

$$\partial_\mu (\sqrt{-g_{00}} \sqrt{g^{(3)}} \rho_0 u^\mu) = 0, \quad (60)$$

while in this gauge  $u^i/u^0$  equals the matter 3-velocity with respect to the normals to the constant- $K$  hypersurfaces. The velocity four-vector  $u^\mu$  can be written in the form

$$u^\mu = \left[ \frac{1 + v_h^2}{-g_{00}} \right]^{\frac{1}{2}} \delta_0^\mu + u^i \delta_i^\mu, \quad (61)$$

where  $v_h^2 := g_{ij} u^i u^j$  equals to first order the square of the velocity three-vector  $u^i/u^0$ , and we used that  $u^\mu u_\mu = -1$ . By substituting expression (61) into the equation of motion (60), we find

$$\frac{\partial}{\partial t} \left[ \left(1 + \frac{1}{2} v_h^2\right) \sqrt{g^{(3)}} \rho_0 \right] + \frac{\partial}{\partial x^i} \left[ \sqrt{-g_{00}} \sqrt{g^{(3)}} \rho_0 u^i \right] = 0, \quad (62)$$

to first order. Using equation (62) and the definition of the spatial average (8), we obtain

$$\begin{aligned} & \lim_{\ell \rightarrow \infty} \frac{\partial}{\partial t} \langle (1 + \frac{1}{2} v_h^2) \rho_0 \rangle(\ell) = \\ & - \lim_{\ell \rightarrow \infty} \left( \frac{\partial}{\partial t} \ln \tilde{N}(x, \ell) \right) \langle (1 + \frac{1}{2} v_h^2) \rho_0 \rangle(\ell) \\ & - \lim_{\ell \rightarrow \infty} N^{-1}(\ell) \int_{\Sigma} dx' \frac{\partial}{\partial x^i} \sqrt{-g_{00}} \sqrt{g^{(3)}} \rho_0 u^i \theta(\ell - \Delta s(x, x')), \end{aligned} \quad (63)$$

where  $\tilde{N}(\ell)$  denotes the dimensionless quotient of  $N(\ell)$ , and a constant with the dimension of a 3-volume. The second term on the right-hand side of equation (63) vanishes due to Gauß's theorem. Combining the remaining terms in equation (63) yields,

$$\lim_{\ell \rightarrow \infty} \frac{\partial}{\partial t} \ln \langle (1 + \frac{1}{2} v_h^2) \rho_0 \rangle(\ell) = - \lim_{\ell \rightarrow \infty} \frac{\partial}{\partial t} \ln \tilde{N}(x, \ell). \quad (64)$$

By integrating equation (64), it follows that

$$\langle (1 + \frac{1}{2}v_h^2)\rho_0 \rangle(t) \propto \frac{1}{\tilde{N}(x, \ell)} \propto \frac{a^3(t_0)}{a^3(t)}, \quad (65)$$

where we used the gauge condition (34). Formula (65) shows that the rest-frame energy density  $\rho_0$ , when integrated over a spatial volume element on  $\Sigma(t)$  which is comoving with the matter flow, is not conserved for a pressureless fluid, while  $\rho_0(1 + \frac{1}{2}v_h^2)$  is conserved to first order.

The spatial average of the energy perturbation  $\delta\rho$  is obtained by expanding equation (57) for  $T^{00}$  to first order, where we use (61) and the gauge condition (31). We find

$$\langle \rho \rangle(t) = \langle (1 + v_h^2)\rho_0 \rangle(t), \quad (66)$$

which combines with equation (65),

$$\langle \rho \rangle(t) = \bar{\rho}(t) + \langle \frac{1}{2}\bar{\rho}v_h^2 \rangle(t), \quad (67)$$

to first order, where we used that  $\langle \rho_0 \rangle(t_0)$  equals  $\bar{\rho}(t_0)$  when perturbations vanish at time  $t_0$ . Indeed, the lowest order contribution to the averaged energy density (66), is given by the sum of the averaged rest-mass of the fluid, and the (nonrelativistic) kinetic energy of the fluid. Since  $v_h^2$  is of the order of  $\epsilon$ , the lowest order correction to the averaged energy perturbation is typically small in the observed universe, but nevertheless significant in the sense of the ambiguity which is related to the freedom of choosing a gauge and an averaging operation (see section 3).

It is interesting to note that there exists a simple relation due to Irvine and Layzer (see e.g., [12]) which relates  $W := 2\pi G\bar{\rho}J_2$ , where  $J_2$  is defined by equation (55), and the energy due to the peculiar velocity  $L := \langle \frac{1}{2}\bar{\rho}v_h^2 \rangle$ . For a pressureless fluid and nonrelativistic motions, it can be shown that  $\frac{\partial}{\partial t}(aW - aL) = L\dot{a}$ , which, assuming that the universe departs from small values of  $J_2$  and  $L$  and relaxes to a nearly time independent bound state at late times, implies the Newtonian virial theorem  $L = W/2$ .

Note that our result differs from a result derived by Futamase (see [1] and [2]), where one finds a peculiar velocity contribution to the averaged energy density which is exactly twice as large as our result (66). This result seems to be based on the erroneous assumption that the integral of rest-frame energy density over a spacelike hypersurface is time independent (this is only true in a gauge where  $v^i - B^i$  vanishes). In this case, equation (57) yields an averaged energy perturbation which is twice the result (66). However, this result violates continuity of the scale factor at the right-hand side of equation (40) when rest-mass is instantaneously and homogeneously converted into kinetic energy or vice versa.

### 4.3 The squared shear contribution

In this subsection we will evaluate the contribution of the term

$$\langle \hat{K}_{ij}\hat{K}^{ij} \rangle, \quad (68)$$

in the averaged constraint equations (40), for scalar and vector perturbations. The scalar and vector part of  $\hat{K}_{ij}$  are coupled to the matter current by the constraint equation (37), which takes the form,

$$\hat{K}_{i;j}^j = 8\pi G J_i, \quad (69)$$

when evaluated in the constant- $K$  gauge. The matter current  $J_i$  is defined by expression (38), and can be expanded as

$$J_i = (\bar{\rho} + \bar{P}) \sum_n \left[ v_{hn}^{(0)} Q_{ni}^{(0)} + v_{hn}^{(1)} Q_{ni}^{(1)} \right], \quad (70)$$

to first order, where  $v_{hn}^{(0)}$  and  $v_{hn}^{(1)}$  denote the scalar and vector components of the velocity three-vector of the matter with respect to the normals to the constant- $K$  hypersurfaces,  $Q_{ni}^{(0)} := -k^{-1} Q_{n|i}$ , and  $\bar{\rho}$  and  $\bar{P}$  denote the background energy and pressure density.

By substituting the traceless part of the extrinsic curvature tensor (29), and the expansion (70) for  $J_i$ , into the constraint equation (69), we obtain

$$\frac{2}{3} (k_n - 3\mathbf{k}/k_n) [\dot{H}_{Tn}^{(0)} - k_n B_n^{(0)}] = aG(\bar{\rho} + \bar{P}) v_{hn}^{(0)} \quad (71)$$

for scalar perturbations, and

$$\frac{1}{2} (k_n - 2\mathbf{k}/k_n) [\dot{H}_{Tn}^{(1)} - k_n B_n^{(1)}] = aG(\bar{\rho} + \bar{P}) v_{hn}^{(1)} \quad (72)$$

for vector perturbations. Expressions (71) and (72) yield expressions for the scalar and vector traceless part of the extrinsic curvature tensor (29), in terms of the matter velocity, which can be used to evaluate the scalar and vector contribution to expression (68). For scalar perturbations we find,

$$\langle \hat{K}_{ij}^{(0)} \hat{K}^{(0)ij} \rangle = \frac{3}{2} a^2 G^2 (\bar{\rho} + \bar{P})^2 \sum_n \frac{v_{hn}^{(0)2}}{(k_n - 3\mathbf{k}/k_n)^2}, \quad (73)$$

and for vector perturbations,

$$\langle \hat{K}_{ij}^{(1)} \hat{K}^{(1)ij} \rangle = 2a^2 G^2 (\bar{\rho} + \bar{P})^2 \sum_n \frac{v_{hn}^{(1)2}}{(k_n - 2\mathbf{k}/k_n)^2}. \quad (74)$$

The coupling between the matter current and the shear of the normals to the constant- $K$  hypersurfaces, can be interpreted as the ‘frame dragging’ effect which occurs in the presence of moving matter (e.g., as in the region around a rotating black hole). It follows from expressions (73) and (74), taking into account the normalizations of the scalar and vector modes (see expression (15)), that the matter current and  $\hat{K}_{ij}$  couple with different strength for scalar and vector perturbations. Furthermore, the strength of the coupling vanishes proportional to  $k_n^{-1}$  when  $k_n \rightarrow \infty$ . Since  $v_{hn}^2 = O(\epsilon)$ , when velocity perturbations are generated by density perturbations at late times, it follows that expressions (73) and (74) contribute to the averaged constraint equations (40) by a term

of order  $\epsilon\kappa^2$ , which is negligible compared to the leading order kinetic energy contribution discussed in section (4.2) for perturbations at length scales much smaller than the Hubble radius.

However, for perturbations at arbitrary large length scales, the strength of the coupling grows proportional to  $\delta^{-1}$  when  $\delta \downarrow 0$ , where  $\delta := k_n^2 - 3\mathbf{k}$  for scalar perturbations and  $\delta := k_n^2 - 2\mathbf{k}$  for vector perturbations. Note that since  $k_n$  must be real for bounded solutions, the limit  $\delta \downarrow 0$  does not exist when  $\mathbf{k} < 0$ , and the limit  $\delta \downarrow 0$  does not exist when  $\mathbf{k} > 0$  since  $k_n$  takes only discrete values in this case.

Note that the divergent coupling between the metric and the matter velocity for  $\delta \downarrow 0$  and  $\mathbf{k} = 0$ , is unrelated to the dynamics of the matter and metric at small scales and late times, since perturbations for which  $\delta \ll 1$  are typically larger than the Hubble radius, and must have a primordial origin.

A natural question which arises is whether the divergence in equations (73) and (74) for  $\delta \downarrow 0$  can be purely attributed to a large warping of the constant- $K$  hypersurfaces, which can be removed by choosing another gauge. Indeed, it follows from expressions (29) and (21) that the scalar part of  $\hat{K}_j^i$  can be set equal to zero, by a temporal gauge transformation with  $T = k^{-2}[\dot{H}_{Tn}^{(0)} - k\dot{B}_n^{(0)}]$ , but according to expressions (49), (22) and (23), the *intrinsic* spatial curvature does diverge when  $\delta \downarrow 0$  in this gauge. Furthermore, due to expression (24), the vector part of  $\hat{K}_j^i$  is gauge invariant, and the divergence in equation (74) is therefore independent of the choice of time-slicing. From the point of view of the matter, the most natural choice of gauge is a comoving time-orthogonal gauge, which is defined by the condition that the spatial coordinates are comoving with the normals to the constant- $t$  hypersurfaces (i.e.,  $B^{(0)} = B^{(1)} = 0$ ), and the scalar part of the matter velocity with respect to the normals to the constant- $t$  hypersurfaces vanishes (i.e.,  $v^{(0)} - B^{(0)} = v^{(0)} = 0$ ). According to expressions (21) and (25), a gauge transformation from the constant- $K$  gauge to a comoving time-orthogonal gauge is generated by  $T = k^{-1}v_{nn}^{(0)}$ . In this gauge, the scalar part of the shear of the *matter* coincides with the scalar part of  $\hat{K}_j^i$ . By transforming equation (71) from the constant- $K$  gauge to a comoving gauge, we find that the infra-red divergence of the scalar part of  $\hat{K}_j^i$  has the same strength in both gauges, and its presence is therefore related to the presence of shearing matter. At first sight, a divergence of the shear of the matter for  $\delta \downarrow 0$  seems to be inconsistent with the smallness of the velocity perturbations which are the source of the metric perturbations. There is no real inconsistency however, since the matter velocity perturbation is gauge dependent, and it might therefore be anomalously small in the constant- $K$  gauge, without being in conflict with large matter shear perturbations. These observations show that the divergence in equations (73) and (74) is of a physical nature.

The absence of FLRW solutions of the constraint equations (69) when homogeneous vector perturbations of the matter velocity are present, might seem peculiar, since solutions of the Einstein equations correspond to stable points of the action. At this point we should recall that we have limited our scope to FLRW background spacetimes, which are by definition spatially homogeneous and isotropic. In the presence of homogeneous matter velocity perturbations, our spacetime is no longer isotropic in the averaged sense, and there is no



FLRW background solution which is everywhere close to our perturbed space-time. A satisfactory description of homogeneous velocity perturbations about FLRW, requires the inclusion of background solutions which are homogeneous but not necessarily isotropic, and which include FLRW as a special case. These solutions are given by the Bianchi models of type V and VII<sub>h</sub>, which include FLRW with  $\mathbf{k} = -1$  as a special case, and type VII<sub>0</sub> which includes FLRW with  $\mathbf{k} = 0$  (see e.g., [14] – [16]).

#### 4.4 The averaged expansion

By substituting the expressions for the averaged curvature perturbation and the averaged energy density, which were derived in the previous subsections 4.1, 4.2 and 4.3, into the averaged constraint equation (40), we obtain,

$$\frac{\dot{a}^2}{a^4} = \frac{8\pi}{3}G\bar{\rho} - \frac{\mathbf{k}}{a^2} + \frac{1}{3}\Lambda + \frac{8\pi G}{6}\langle\bar{\rho}v_h^2\rangle - \frac{32\pi^2}{3}(G\bar{\rho})^2J_2 \quad (75)$$

$$+ \text{g.w.} + O(\epsilon^2, \epsilon\kappa^2, \epsilon^3/\kappa^2),$$

where  $J_2$  is defined by equation (55), and the term g.w. denotes the contribution due to gravitational waves (see subsection 4.1.3). We see that the averaged constraint equation (75) takes the form of the standard Friedmann equation, plus a contribution due to the peculiar velocity of the matter, and a contribution due to the averaging of scalar and tensor metric perturbations. Let us now determine the magnitudes of the different contributions on the right-hand side of equation (75) by means of the observational values for  $\bar{\rho}$  and  $J_2$ . Estimates from the Lick and CfA catalogs [17] [18] value  $J_2 \approx 200h^{-2} \text{ Mpc}^2$ , and  $\bar{\rho} \approx 1.88 \times 10^{-29}h^2\Omega \text{ g cm}^{-3}$ , where  $h$  is a dimensionless factor which expresses the uncertainty in the value of the Hubble parameter  $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ , and  $h$  is believed to be between 0.5 and 0.85. Inserting these values in the different terms on the right-hand side of equation (75), one finds,

$$\frac{8\pi G}{3}\bar{\rho} = 1.14 \times 10^{-35}h^2\Omega\text{s}^{-2}, \quad (76)$$

$$\frac{32\pi^2}{3}(G\bar{\rho})^2J_2 \sim 1.0 \times 10^{-39}h^2\Omega^2\text{s}^{-2}, \quad (77)$$

and

$$\frac{8\pi G}{6}\langle\bar{\rho}v_h^2\rangle \sim 1.3 \times 10^{-40}h \Omega\text{s}^{-2}, \quad (78)$$

where we used the relation  $v \sim (3\pi G\bar{\rho}J_2)^{1/2}$  (see section 4.2). According to equations (76)–(78), and the constraint equation (75), the matter induced metric inhomogeneities act as a very small negative correction to the averaged energy density, equal to about  $1.0 \times \Omega$  part in  $10^4$ , while the back-reaction due to the peculiar velocity of the matter acts as a positive correction to the averaged energy density, equal to about 1.2 parts in  $10^5$ . The small negative correction to the averaged energy density leads to a slight overestimation of the age of the universe  $t_0 = \frac{2}{3}H_0^{-1}$  assuming that  $\Omega = 1$ , equal to about 5 parts in  $10^5$ .

## 5 Comparison with previous work

The work on this paper started as a correction of the derivation by Futamase in [1] [2] on the points of the treatment of the gauge freedom (see section 3) and the choice of the averaging operation (see section 2). The paper was also inspired as an attempt to address the fundamental ambiguity which enters the calculation of any averaged metric through the freedom of choosing an averaging operation.

In a recent independent paper by Russ *et al* [21], the back-reaction due to density perturbations was calculated by using the relativistic Zel'dovich approximation [22] in a comoving gauge. The expression derived by Russ *et al* for the back-reaction due to matter density perturbations agree in sign, but is roughly an order of magnitude larger than the result derived in this paper. Furthermore, a possible effect due to vorticity of the matter was ignored in that paper. It should be noted that direct comparison between the results by Russ *et al* and the results derived in this paper, is nontrivial due to the fact that the gauges used in the two papers are not related by a first order gauge transformation. Namely, a gauge transformation from the constant- $K$  gauge to the comoving synchronous gauge requires  $\dot{L}_n = -v_{hn}$  due to equation (25), and  $v_{hn} = O(\epsilon^{1/2})$  since  $v_{hn}^2 = O(\epsilon)$  when velocity perturbations are generated by density perturbations at late times. By working in a constant- $K$  gauge, we avoided the problem of a breakdown of the perturbative expansion which occurs in the comoving gauge (namely, since metric and matter density perturbations are of the same magnitude in a comoving gauge, metric perturbations get typically large at late times, even though the perturbations in the intrinsic geometry are generally small in the observed universe).

Finally, we mention the paper by Buchert and Ehlers [20], where one integrates the Raychaudhuri equation over a spatial hypersurface in a Newtonian background, and a globally vanishing correction to the averaged expansion was found. Although the Raychaudhuri equation is also valid in GR, the Newtonian approximation enters the calculation where the correction to the averaged expansion is expressed in terms of a boundary term, which accounts for the difference between the Newtonian result and the nontrivial correction (76) to the averaged energy density derived in this paper.

## 6 Conclusions

We derived the generic linearized averaging operation for metrics starting from the requirement that unperturbed FLRW is a stable fixed point of the averaging operation. By a gauge invariant approach, we eliminated unphysical degrees of freedom in our problem, and we explicitified the fundamental ambiguities which are related to the freedom of fitting the averaged spacetime to the inhomogeneous spacetime. The leading order nontrivial corrections to the standard Friedmann equation are expressed in terms of the power spectrum of the matter, and the effect is calculated quantitatively by means of the observational data. The dominant correction to the averaged expansion is caused by the back reaction of matter density perturbations, and leads to a slower expansion rate and an overestimation of the age of the universe by approximately 5 parts in

<sup>10</sup><sub>5</sub>. The back-reaction of velocity perturbations, including vortical motion of the matter, appears to be negligible at small length scales. However, it was shown that the back-reaction of velocity perturbations can be significant in the large wavelength limit.

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## 8 Appendix A

In this appendix, we will discuss the relation between the volume element in the hypersurface integral, and gauge invariance at second and higher order in the expansion parameter of the gauge transformation.

Let  $\phi$  be a one parameter group of diffeomorphisms  $\phi : \mathbf{R} \times \Sigma \rightarrow \Sigma$ , which is defined by the condition that  $\phi_{\lambda=0}$  is the identity, and the curves  $\phi_\lambda(p)$  are integral curves of a vector field  $\xi$  in  $\Sigma$  (see e.g. [19] and [10] for the mathematical details which are involved). A gauge is specified by choosing a mapping between points  $p$  in  $\Sigma$ , and points  $\bar{p}$  in  $\bar{\Sigma}$ . Assuming that a choice of gauge has been made, then a one parameter *group* of gauge choices is obtained by mapping the points  $\phi_\lambda(p)$  in  $\Sigma$  to points  $\bar{p}$  in  $\bar{\Sigma}$ , for all  $\lambda \in \mathbf{R}$  (the more generic case of a one parameter *family* of mappings of points in the background and the perturbed spacetime, is discussed in [10], but there is no need to introduce this complication in the derivation which follows).

Let us consider a scalar function  $q(x)$ , which lives in  $\Sigma$  (such that its value in a point  $p$  in  $\Sigma$  is fixed, while its value in a point  $\bar{p}$  in  $\bar{\Sigma}$  depends on the choice of gauge). The spatial average and the hypersurface integral of  $q(x)$ , are related by

$$\langle q \rangle = \langle \langle q (g^{(3)}/g^B)^{\frac{1}{2}} \rangle \rangle, \quad (79)$$

where we used the definition (13). The integrand at the right-hand side of equation (79) is gauge dependent, and may be expanded in powers of  $\lambda$  about  $\lambda = 0$ , i.e.,

$$q(g^{(3)}/g^B)^{\frac{1}{2}}(\lambda, \bar{p}) = \sum_{k=0}^{k=\infty} \frac{\lambda^k}{k!} \mathcal{L}_\xi^k q(g^{(3)}/g^B)^{\frac{1}{2}}, \quad (80)$$

where  $\mathcal{L}_\xi^k$  denotes the  $k$ -th order Lie derivative with respect to  $\xi$ , evaluated in  $\bar{p}$ . By substituting the expansion (80) in the integrand at the right-hand side of equation (79), we obtain

$$\langle q \rangle(\lambda) - \langle q \rangle(0) = \sum_{k=1}^{k=\infty} \frac{\lambda^k}{k!} \langle \langle \mathcal{L}_\xi^k q(g^{(3)}/g^B)^{\frac{1}{2}} \rangle \rangle. \quad (81)$$

For  $k = 1$ , the contribution to the right-hand side of equation (81) is evaluated using

$$\mathcal{L}_\xi q = \xi^i q_{;i} \quad (82)$$

and

$$\mathcal{L}_\xi (g^{(3)}/g^B)^{\frac{1}{2}} = \xi^i_{;i} (g^{(3)}/g^B)^{\frac{1}{2}}, \quad (83)$$

where  $;$  denotes covariant differentiation with respect to  $g_{ij}$ . Combining equations (82) and (83) yields,

$$\mathcal{L}_\xi q (g^{(3)}/g^B)^{\frac{1}{2}} = (q\xi^i)_{;i} (g^{(3)}/g^B)^{\frac{1}{2}}, \quad (84)$$

and

$$\langle\langle (q\xi^i)_{;i} (g^{(3)}/g^B)^{\frac{1}{2}} \rangle\rangle = 0, \quad (85)$$

due to Gauss's theorem. The  $k = 2$  contribution to the right-hand side of equation (81) is obtained by making the substitution  $q \rightarrow (q\xi^i)_{;i}$  in expression (84), and for arbitrary  $k \in \mathbf{Z}^+$  the same result follows by induction. Since the terms at the right-hand side of equation (81) vanish for all  $k$ , we established that the spatial average of a scalar function  $q$  is gauge invariant to arbitrary order in the expansion parameter  $\lambda$ . Applying the same analysis as above to the hypersurface integral of a scalar field  $q(x)$ , we find,

$$\langle\langle q \rangle\rangle(\lambda) - \langle\langle q \rangle\rangle(0) = \sum_{k=1}^{k=\infty} \frac{\lambda^k}{k!} \langle\langle \mathcal{L}_\xi^k q \rangle\rangle, \quad (86)$$

which depends on  $\lambda$ , due to equation (82), unless  $q$  is a constant on  $\Sigma$ . A similar derivation, where we reverse the roles of the spacetimes  $\Sigma$  and  $\bar{\Sigma}$ , shows that the hypersurface integral of a scalar field  $\bar{q}(\bar{x})$  which lives in  $\bar{\Sigma}$  is gauge invariant, while the spatial average of  $\bar{q}(\bar{x})$  is gauge invariant iff  $\bar{q}(\bar{x})$  is constant in  $\bar{\Sigma}$ . It follows from these observations that the spatial average of a perturbation  $\delta q := q(x) - \bar{q}(\bar{x})$  is gauge invariant iff  $\bar{q}(\bar{x})$  is constant on  $\bar{\Sigma}$ .

Note that Futamase in [1] uses the hypersurface integral as a spatial averaging operation in the calculation of second order effects, while he does not consistently fix a gauge in these papers (namely, he assumes a comoving synchronous gauge, *and* constant expansion on the hypersurfaces of constant time coordinate).

## 9 Appendix B

In this appendix we derive the decomposition of the generic linearized averaging operation

$$\hat{A}^\infty := \lim_{n \rightarrow \infty} \hat{A}^{(1)n} \delta g_{\mu\nu}, \quad (87)$$

in terms of the spatial average  $\langle\delta g_{\mu\nu}\rangle(t)$ , which is uniquely defined. Note that the existence of the limit (87) implies that

$$\hat{A}^{(1)} \delta g_{\mu\nu}^* = \delta g_{\mu\nu}^* \quad (88)$$

for arbitrary spatially homogeneous and isotropic perturbations  $\delta g_{\mu\nu}^*$  (up to the freedom of diffeomorphisms acting at either side of equation (88)).

Without loss of generality, a spatially homogeneous and isotropic perturbation  $\delta g_{\mu\nu}^*$  about  $\bar{\mathbf{S}}$  can be written in the form

$$\delta g_{\mu\nu}^* = \phi_1(\bar{t})\bar{n}_\mu\bar{n}_\nu + \phi_2(\bar{t})\bar{h}_{\mu\nu}, \quad (89)$$

where  $\bar{h}_{\mu\nu} := g_{\mu\nu}^B + \bar{n}_\mu\bar{n}_\nu$ , and  $\bar{n}_\mu$  denotes the timelike future directed vector in  $\bar{\mathbf{S}}$  which is orthogonal to  $\bar{\Sigma}$ , and which is normalized with respect to the background metric  $g_{\mu\nu}^B$ , and  $\phi_1(\bar{t})$  and  $\phi_2(\bar{t})$  are arbitrary functions of  $\bar{t}$ .

When we substitute expression (89) for  $\delta g_{\mu\nu}^*$  and expression (3) for  $\hat{A}^{(1)}$  into condition (88), we obtain,

$$\begin{aligned} \int dt' d^3x' \left( \phi_1(t')\bar{n}_\rho\bar{n}_\sigma + \phi_2(t')\bar{h}_{\rho\sigma} \right) f_{\mu\nu}^{\rho\sigma}(x, x') \\ = \phi_1(t)\bar{n}_\mu(x)\bar{n}_\nu(x) + \phi_2(t)\bar{h}_{\mu\nu}(x), \end{aligned} \quad (90)$$

for arbitrary functions  $\phi_1(t)$  and  $\phi_2(t)$ . Equation (90) holds for arbitrary  $\phi_1(t)$  and  $\phi_2(t)$  iff

$$\int_{\bar{\Sigma}} d^3x' \bar{n}_\rho(x')\bar{n}_\sigma(x') f_{\mu\nu}^{\rho\sigma}(x, x') = \delta(t' - t)\bar{n}_\mu(x)\bar{n}_\nu(x), \quad (91)$$

and

$$\int_{\bar{\Sigma}} d^3x' \bar{h}_{\rho\sigma}(x') f_{\mu\nu}^{\rho\sigma}(x, x') = \delta(t' - t)\bar{h}_{\mu\nu}(x). \quad (92)$$

Expression (91) shows that  $f_{\mu\nu}^{\rho\sigma}(x, x')$  is proportional to a delta distribution  $\delta(t - t')$ . It follows from this observation that  $\hat{A}^{(1)}$  can be naturally defined in terms of a linearized *spatial* averaging operation  $\hat{A}_s^{(1)}$ , i.e.,

$$\hat{A}^{(1)}\delta g_{\mu\nu} = \hat{A}_s^{(1)}(t)\delta g_{\mu\nu}, \quad (93)$$

where  $\hat{A}_s^{(1)}$  is defined by,

$$\hat{A}_s^{(1)}\delta g_{\mu\nu} = \int_{\bar{\Sigma}(t)} d^3x' f_{\mu\nu}^{\rho\sigma}(t, x^i, x^{i'}) \delta g_{\rho\sigma}(x^{i'}), \quad (94)$$

and

$$f_{\mu\nu}^{\rho\sigma}(t, x^i, x^{i'}) := \int_{\Delta t'} dt' f_{\mu\nu}^{\rho\sigma}(t, t', x^i, x^{i'}), \quad (95)$$

and  $\Delta t'$  is chosen such that  $t \in \Delta t'$ . At first sight, the decomposition of the linear averaging operation  $\hat{A}^{(1)}$  in terms of a spatial averaging operation which is defined on a collection of spatial hypersurfaces might be surprising, since the choice of a collection of spatial hypersurfaces  $\bar{\Sigma}(t)$  in  $\mathbf{S}$  is gauge dependent. It will be shown in section (3) that although the choice of  $\bar{\Sigma}(t)$  in  $\mathbf{S}$  is gauge dependent, the linearized spatial averaging operation (93) is to first order gauge independent.

Assuming that the limit (5) exists, then by substituting expression (93) into expression (87) one finds that the limit

$$\langle \delta g_{\mu\nu} \rangle := \lim_{n \rightarrow \infty} \hat{A}_s^{(1)n} \delta g_{\mu\nu}, \quad (96)$$

exists. We will show that the limiting spatial averaging operation which is defined by equation (96) is universal.

Expression (94) and the definition (96) imply that,

$$\langle \delta g_{\mu\nu} \rangle := \lim_{n \rightarrow \infty} \int_{\Sigma(t)} d^3 x' f_{\mu\nu}^{n\rho\sigma}(t, x^i, x'^i) \delta g_{\rho\sigma}(x'^i), \quad (97)$$

where  $f_{\mu\nu}^{n\rho\sigma}$  is defined in terms of  $f_{\mu\nu}^{\rho\sigma}$  by induction over  $n$ ,

$$f_{\mu\nu}^{n\rho\sigma}(x^i, x'^i) = \int_{\Sigma(t)} d^3 q f_{\alpha\beta}^{\rho\sigma}(t, x'^i, q^i) f_{\mu\nu}^{n-1\alpha\beta}(t, x^i, q^i), \quad (98)$$

and  $f_{\mu\nu}^{1\rho\sigma} := f_{\mu\nu}^{\rho\sigma}$ . Let us now try to determine the limit

$$f_{\mu\nu}^{\infty\rho\sigma} := \lim_{n \rightarrow \infty} f_{\mu\nu}^{n\rho\sigma}. \quad (99)$$

An explicit calculation of  $f_{\mu\nu}^{\infty\rho\sigma}$ , using the definition (98) for  $f_{\mu\nu}^{n\rho\sigma}$  and starting with arbitrary realizations for  $f_{\mu\nu}^{\rho\sigma}$ , would be quite cumbersome, but fortunately it appears that the symmetries of the background spacetime  $\bar{S}$ , and the stability condition (5) determine  $f_{\alpha\beta}^{\infty\rho\sigma}$  completely.

Recall that we required that the limit (5) converges to a spatially homogeneous and isotropic metric perturbation for arbitrary perturbations  $\delta g_{\mu\nu}$ , which implies that

$$\langle \delta g_{\mu\nu} \rangle(x^i) = \int_{\Sigma(t)} d^3 x' f_{\mu\nu}^{\infty\rho\sigma}(t, x^i, x'^i) \delta g_{\rho\sigma}(x'^i) = \delta g_{\mu\nu}^*, \quad (100)$$

for all  $x$ , where we used expression (93) and  $\delta g_{\mu\nu}^*$  has the form (89). If expression (100) holds for arbitrary perturbations  $\delta g_{\rho\sigma}(x'^i)$ , it also holds for arbitrary perturbations  $\delta g_{\rho\sigma}(x'^i + c^i)$ , where  $c^i \in \mathbf{R}$ . By absorbing the constants  $c^i$  into the coordinates  $x^i$ , one finds that expression (100) remains unchanged under the substitution

$$f_{\mu\nu}^{\infty\rho\sigma}(x^i, x'^i) \rightarrow f_{\mu\nu}^{\infty\rho\sigma}(x^i, x'^i - c^i). \quad (101)$$

Furthermore, since the right-hand side of equation (100) is spatially homogeneous by requirement, we find that the left-hand side of equation (100) must be also invariant under the substitution

$$f_{\mu\nu}^{\infty\rho\sigma}(x^i, x'^i) \rightarrow f_{\mu\nu}^{\infty\rho\sigma}(x^i + d^i, x'^i), \quad (102)$$

where  $d^i \in \mathbf{R}$  is arbitrary. Since equation (100) is invariant under (101) and (102) for arbitrary perturbations  $\delta g_{\mu\nu}$ , we conclude that  $f_{\mu\nu}^{\infty\rho\sigma}$  is (up to the freedom of performing diffeomorphisms) constant on  $\Sigma$  when regarded as a distribution (i.e., neglecting sets of Lebesgue measure zero). Furthermore, since equation (100) holds for arbitrary  $\delta g_{\mu\nu}$ , the distribution  $f_{\mu\nu}^{\infty\rho\sigma}(x, x')$  must be proportional to a tensor of the form (89) in the point  $x$ , thereby fixing the  $\mu\nu$  dependent part of  $f_{\mu\nu}^{\infty\rho\sigma}$ . We may therefore write

$$f_{\mu\nu}^{\infty\rho\sigma}(x, x') = g_1^{\rho\sigma}(x') \bar{n}_\mu(x) \bar{n}_\nu(x) + g_2^{\rho\sigma}(x') \bar{h}_{\mu\nu}(x), \quad (103)$$

where  $g_1^{\rho\sigma}(x')$  and  $g_2^{\rho\sigma}(x')$  are spatially homogeneous tensor densities in  $x'$ , and we used expression (89) for  $g_{\mu\nu}^*$ .

A similar argument, using the invariance of equation (100) under the group of spatial rotations (using that  $\mathcal{F}$  does not explicitly depend on  $x$ ), shows that the bi-tensor density  $f_{\mu\nu}^{\infty\rho\sigma}(x, x')$  is isotropic with respect to the indices  $\sigma$  and  $\rho$ , which implies that the tensor densities  $g_1^{\rho\sigma}(x')$  and  $g_2^{\rho\sigma}(x')$  in expression (103) are of the form,

$$g_1^{\rho\sigma}(x') = \alpha_1 \sqrt{g^{(3)}} \bar{n}^\rho(x') \bar{n}^\sigma(x') + \alpha_2 \sqrt{g^{(3)}} \bar{h}^{\rho\sigma}(x') \quad (104)$$

and

$$g_2^{\rho\sigma}(x') = \alpha_3 \sqrt{g^{(3)}} \bar{n}^\rho(x') \bar{n}^\sigma(x') + \alpha_4 \sqrt{g^{(3)}} \bar{h}^{\rho\sigma}(x') \quad (105)$$

where  $g^{(3)}$  denotes the real space volume element, which follows from requiring spatial gauge invariance at higher orders (see appendix A), and the factors  $\alpha_n$  ( $n \in \{1, 2, 3, 4\}$ ) are constant on  $\Sigma$ . Substituting expressions (104) and (105) in expression (103) yields

$$f_{\mu\nu}^{\infty\rho\sigma}(x, x') = \sqrt{g^{(3)}} (\alpha_1 \bar{n}^\rho(x') \bar{n}^\sigma(x') \bar{n}_\mu(x) \bar{n}_\nu(x) + \alpha_4 \bar{h}^{\rho\sigma}(x') \bar{h}_{\mu\nu}(x)), \quad (106)$$

where we used expressions (91) and (92) to show that the terms proportional to  $\alpha_2$  and  $\alpha_3$  vanish.

By substituting expression (106) for  $f_{\mu\nu}^{\infty\rho\sigma}$  into condition (100), where we set  $\delta g_{\rho\sigma}$  equal to  $\delta g_{\mu\nu}^*$  defined by expression (89), we find that the constants  $\alpha_1$  and  $\alpha_4$  in expression (106) must satisfy the condition

$$\int_{\Sigma(t)} d^3x' \sqrt{g^{(3)}} \alpha_1 = 3 \int_{\Sigma(t)} d^3x' \sqrt{g^{(3)}} \alpha_4 = 1. \quad (107)$$

Expression (107) shows that the constants  $\alpha_1$  and  $3\alpha_4$  are equal to  $(\text{volume}(\Sigma))^{-1}$  when  $\Sigma$  is closed, while in the case when  $\Sigma$  is open,  $\alpha_1$  and  $\alpha_4$  are defined in a distributional sense by condition (107), and by the condition that  $\alpha_1$  and  $\alpha_4$  are constant on  $\Sigma(t)$ .

By substituting expression (106) into expression (100) we obtain the explicit expression for the spatial average,

$$\langle \delta g_{\mu\nu} \rangle = \int_{\Sigma(t)} d^3x' \sqrt{g^{(3)}} \alpha_1 \quad (108)$$

$$[\bar{n}^\rho(x') \bar{n}^\sigma(x') \bar{n}_\mu(x) \bar{n}_\nu(x) + \frac{1}{3} \bar{h}^{\rho\sigma}(x') \bar{h}_{\mu\nu}(x)] \delta g_{\rho\sigma}. \quad (109)$$

Note that  $\bar{n}^\rho \bar{n}^\sigma \delta g_{\rho\sigma}$  equals the perturbation of  $g_{00}$  in coordinates which are synchronous in the background (i.e., coordinates for which  $g_{\mu 0}^B = -\delta_\mu^0$ ), while  $\bar{h}^{\rho\sigma} \delta g_{\rho\sigma}$  equals the perturbation of the spatial volume element on  $\Sigma$ , to first order.

Summarizing the derivation in this appendix, we showed that the general linearized averaging operation which is a functional of metric perturbations about FLRW, and for which unperturbed FLRW is a stable fixed point, has a unique limit when applied iteratively to perturbed FLRW. Furthermore, we showed that this linearized averaged operation is naturally defined in terms of a spatial averaging operation which works on  $g_{00}$  and the spatial volume perturbation in coordinates which are synchronous in the background.

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